SOME BASIC FEATURES IN MECHANICS
OF INHOMOGENEOUS MATERIALS

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Synopsis

The authors show several aspects of the mechanics of inhomogeneous materials. They point out the necessity of studying the composites in dependence on the ratio of the dispersed, dispersing and liquid phases in the system and propose their classification. They define the functional expressions for the principal elastic constants of two main composite systems, and suggest the possibility of describing materials by a single constant, the absorbent of elasticity, as a general function of the material definition (the composition of the material) and its history.

1. Introduction

The development of a physically justified theory of mechanics of inhomogeneous media should originate from the general but determining features of composite materials. Two fundamental features of composite materials with dispersed particles, which are the main determining factors of their properties if they are considered to be structural systems, are represented by the structure and the presence of internal interphase boundaries. It will be shown that both these factors change expressively but regularly, if the volume ratio of the individual phases is changed.

The main mechanical properties of the materials are undoubtedly their elasticity parameters; therefore, suitable structural models of the composites were found and their elasticity constants were determined by means of a suitable combination of identical properties of the components.

The dependence of the properties of the material on time and the complications due to a great number of incommensurable conventional material constants, however, calls for the introduction of an effective characteristic function of the material; a proposal for the latter is discussed in a later section.

The influence of the interface is an independent and extensive problem and will be treated elsewhere.

2. Structural Definition of Composites

We have already shown in the Comprehensive Abstract (Ref. 1) on the basis of experimental data that there exists a striking qualitative diversity in the properties of materials according to the volume ratios of the phases. The analysis of the behaviour of any composite in several distinct classes is, therefore, unavoidable. It was found that the macro-dispersion structural materials behave differently under external load primarily according to whether the dispersed solid phase is segregated or aggregated. In a system with an aggregated solid phase it becomes immediately significant whether the liquid phase is discontinuous or continuous, in other words, whether the boundary between the solid and liquid phases in the superstructure*[1] is opened or closed*[2].

Thus, the composites can be classified, with the usual risk of simplification, into four fundamental ranges of composition limited by five limiting systems.
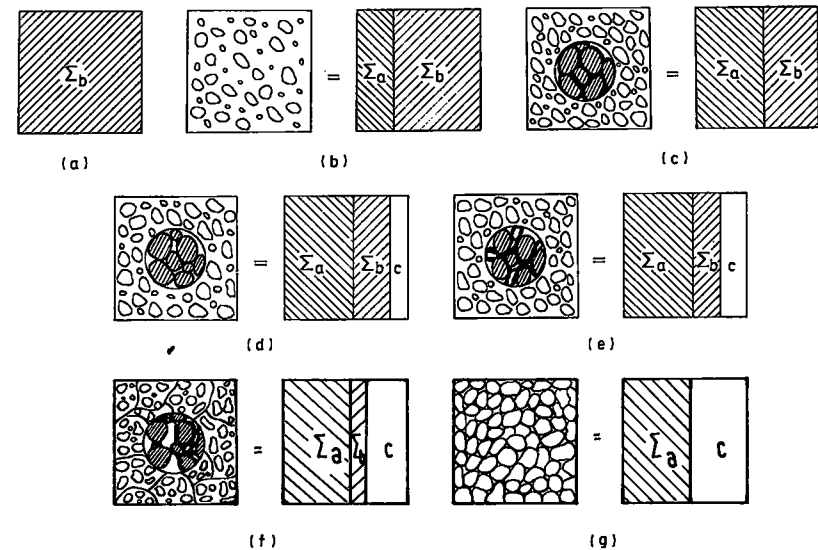

Fig. 1. Schemes of structural systems.

*[1] The superstructure is the structure of the system as opposed to the inherent structure of the individual compounds. The infrastructure is the structure of the dispersed, dispersing and liquid phases.
*[2] It is then significant whether the dispersing phase is sufficient for forming a continuous matrix or not; in the former case the macrodispersed material is rigid, in the latter loose.

In the first range (filled binder) the solid phase b forming the dispersing medium (matrix) predominates. The particles of the dispersed phases a (Fig. 1 (b)) are segregated in the matrix at distances of more than $2\Delta r$, where Δr is the thickness of the coated layer of the matrix on the particle corresponding to the minimum free energy of the system. This range starts from the first limiting system containing no dispersed particles, the superstructure being identical with the infrastructure of the dispersing medium (Fig. 1 (a)), and ends with the second limiting system. The latter is built up of an aggregate of dispersed particles, now at a distance of $2\Delta r$, and of the dispersing medium to form a compact material (Fig. 1 (c)). In the first range the liquid phase is not present in the superstructure (expect for the liquid on the level of the structural systems). In the other ranges the density of the dispersed particles in the system remains constant and maximum and the particles are aggregated.

In the second range (bond filler) the liquid phase c enters into the system of the superstructure to the detriment of the dispersing medium, and exists there in confined spaces. In this range, though rather narrow, the properties of the system change rapidly and considerably (Fig. 1 (c)). With the increasing volume ratio the spaces with the liquid join up gradually forming a continuous infrastructural system of the liquid phase. This determines the third limiting system (Fig. 1 (e)).

In the third range (bond filler) the share of the liquid phase increases further to the detriment of the dispersing medium (Fig. 1 (e)). The superstructural system is porous and the newly created infrastructural system of the liquid phase depends on the ambient medium. The infrastructure of the dispersed phases displays unchanged density up to the fourth limiting system, where the superstructural system begins to disintegrate in clusters due to the lack of a primary dispersing medium; the secondary dispersing medium, the liquid phase, at the most remains among the clusters (Fig. 1 (f)).

In the fourth range the density of the infrastructure of the dispersed phase increases slightly; the superstructural system, having originally the nature of a rigid cohesive material, changes to a loose, non-cohesive material, from a coarse-grained material at first (composed of large clusters of the used structural systems) to a fine-grained one (composed only of the dispersed structural systems). This system is obviously the last, the fifth limiting system when the primary dispersing medium (matrix) fully vanishes (Fig. 1 (g)). If the ambient medium is dry, there is no liquid phase present in this limiting case, and the superstructural system is identical with the existing infrastructural system of the dispersed phase of the aggregate.

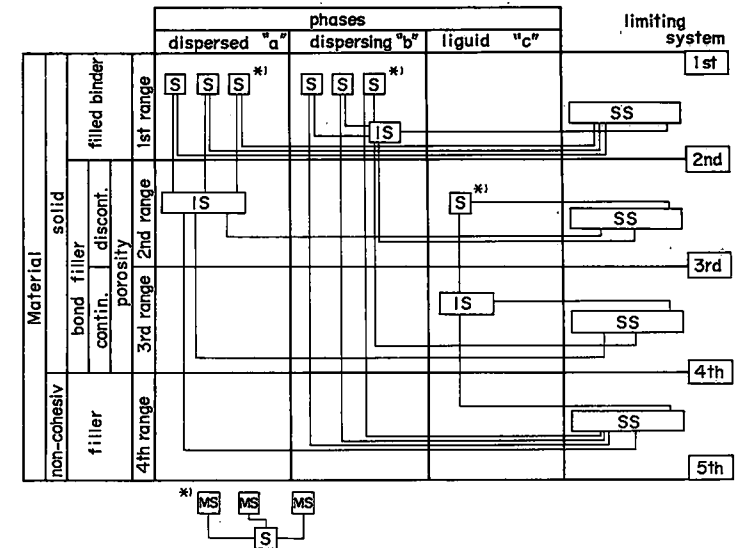The classification and compositions of the superstructural systems

Fig. 2. Classification and compositions of the superstructural systems. MS-microstructure (substance), S-structure, IS-infrastructure, SS-superstructure (of a material, respectively).

of the composites are schematically shown in Fig. 2.

All mechanical and physical properties of the composites at a given time depend on: (1) the properties of the individual phases, (2) the volume proportion of these phases in the system, (3) the free-space content of the dispersed phase, (4) the interaction between the dispersed and the dispersing phases, and (5) the interaction between the superstructure and the ambient medium. Moreover, naturally, they all depend on time factors, $i.e.$, on the age and history of the individual phases, the age and history of the superstructure, the rate of change of the external physical conditions, the loading rate, etc.

The fundamental characteristics of the materials, the constants of elasticity, are presented below for the two main types of composites: the filled binder (the first range) and the bond filler (the second and third ranges).

3. Elastic Constants of the Filled Binder

On formulating the relations between the elastic constants of the included solid phases on the one hand and the solid compact, quasi-homogeneous, quasi-isotropic system (of the filled binder type) from these phases on the other, Dantu (Ref. 2) and Kaplan (Refs. 3, 4) con-

cluded that Young's modulus E of a bi-phase system depends on Young's moduli, E_1 and E_2, and the volume amounts, P and $(1-P)$, of both the phases according to Eqs. (1) and (2):

$$\frac{1}{E} = \frac{P}{E_1} + \frac{1-P}{E_2}, \tag{1}$$

$$E = PE_1 + (1-P)E_2. \tag{2}$$

The possibilities of combining the phase moduli with the modulus of the system were limited by these equations for the case of the neglected or zero Poisson's ratios. The limiting nature of these equations was proved by Paul (Ref. 5) who made use of the minimum strain-work and the minimum potential energy principles. The boundary systems, behaving in complete agreement with the above limiting equations, were demonstrated by Hansen (Refs. 6, 7) as stratified models with the layers of both the phases either perpendicular or parallel to the direction of the acting uniaxial stress, called soft and hard materials, respectively.

Since the real elastic behaviour of the bi-phase system corresponds to a compromise between the limiting formulations, Hirsch (Ref. 8) and Dougill (Ref. 9) combined Eqs. (1) and (2) into a unified Eq. (3):

$$\frac{1}{E} = (1-A)\left(\frac{P}{E_1} + \frac{1-P}{E_2}\right) + \frac{A}{PE_1 + (1-P)E_2}. \tag{3}$$

Experimental tests of the concrete led to a value of $1/2$ for the empirical quantity A, this value being assigned to the quasi-isotropy of the real material, i.e., to the presence of soft and hard materials in equal quantities in it. At this point it should be mentioned that in accordance with the concept of stratified models a value of $2/3$ could result for A for this reason.

The first step (Ref. 10) was to establish the following statement in an unempirical way, exploiting the minimum strain-work principle: An elastic compliance, understood to be a quantity inverse to the Young's modulus, of a real material represents the arithmetical average of both the compliances of the same system behaving like soft material in one case, and hard in the other.

The widening of the stratified model concept was the second step taken. The system with a homogeneous stress-flow density was denoted as the soft material, where all slips, resulting in zero shears and thus in an orthogonal behaviour of the system, appear. The hard material is then represented by a system with a homogeneous strain density where, contrary to the soft material, all the slips are impossible and the shears infinite, which, however, also leads to orthogonal behaviour.

Since every elastic behaviour of a real material can then also be considered orthogonal, representing a compromise between the two extremes, the authors were able to formulate, with respect to Poisson's ratios μ_1 and μ_2 of the phases, the $H\text{-}D\text{-}N$ relations for Young's modulus

E and Poisson's ratio μ of the compact bi-phase system in Eqs. (4) and (5):

$$E = \frac{2}{B} E_1 E_2 C, \tag{4}$$

$$\mu = \frac{1}{B}\{[\mu_2(1-P)E_1 + \mu_1 PE_2]C + E_1 E_2[\mu_1(1+\mu_2)(1-2\mu_2)PE_1 \\ + \mu_2(1+\mu_1)(1-2\mu_1)(1-P)E_2]\}, \tag{5}$$

where

$$B = [(1-P)E_1 + PE_2]C + E_1 E_2[(1+\mu_2)(1-2\mu_2)PE_1 \\ + (1+\mu_1)(1-2\mu_1)(1-P)E_2]$$

and

$$C = [(1+\mu_2)PE_1 + (1+\mu_1)(1-P)E_2][(1-2\mu_2)PE_1 + (1-2\mu_1)(1-P)E_2].$$

4. Elastic Constants of the Bond Filler

In view of the independence of the arrangements of the phases in the compact system, the derived relations lose their purpose in the case of vacating the volume of one phase; there does not exist any possibility of a direct widening of the validity of these relations for the porous quasi-homogeneous quasi-isotropic system.

However, the common porous materials can be divided according to their morphology into two principal groups, each of the groups being represented by a real material with a simplified structure of either a foam or bond filler type. Due to this simplication the number of kinds of orthogonally shaped structural elements in the skeleton of such a system is minimized (as well as the number of dimensional parameters). Since the structural elements of each kind are characterized by a specific mechanical behaviour, the possibility of manipulating the porous system properties from the quasi-phase point of view arises.

In the study of the quasi-homogeneous quasi-isotropic binder-filler composites along the whole range of compositions, a lot of attention was
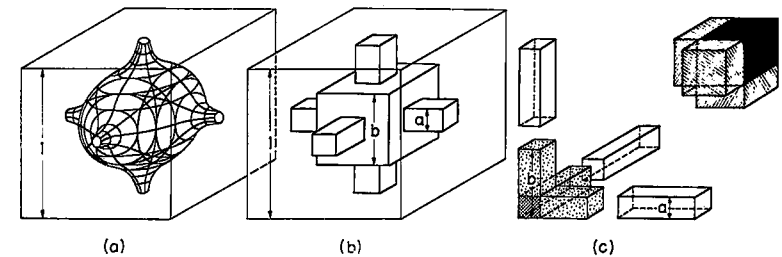

Fig. 3. (a) Volume unit of a permeable material of the bond filler type.
(b) Volume unit simplified into the real system.
(c) Simplified unit divided into the participant structural elements.

paid to the real permeable system of the bond filler type (Fig. 3 (a) to (c)). This approach enabled the authors to formulate the relations for the Young's modulus $\underline{E}_0$ (under external loading, of course) and the Poisson's ratio $\underline{\mu}_0$ as well as the bulk modulus K_{in} under the sole internal pressure of the said permeable system (of the bond filler type) in the dry state according to Eqs. (6) to (8). These quantities depend on the elastic constants E, μ of the homogeneous and isotropic skeleton-solid and on the dimensions a, b of the structural elements related to the unit of the system.

$$\underline{E}_0 = \frac{6}{D} a^2 E, \tag{6}$$

$$\underline{\mu}_0 = \frac{1}{D}(3a\mu + H + J), \tag{7}$$

$$K_{in} = \frac{2}{2-F}\frac{E}{3(1-2\mu)}, \tag{8}$$

where

$$D = 3\left[1 + (1-b)\frac{(1+\mu)(1-2\mu)}{1-\mu}\right] - H + 2J,$$

$$F = \frac{1}{1-2\mu}\left[1 - 2a\mu + (1-b)\frac{(1+\mu)(1-2\mu)}{1-\mu} - H\right],$$

$$H = (1-2\mu)\left\{\frac{a(b-a)}{b}M + \left[(1-b)\frac{2\mu}{1-\mu} - a\right]Q\right\},$$

$$J = (1+\mu)\left\{\frac{a(b-a)}{b}N + \left[(1-b)\frac{\mu}{1-\mu} + a\right]R\right\},$$

$$M = \frac{a(1+\mu) - 2b\mu Q}{2a\mu - b(1+\mu)},$$

$$N = \frac{a(1-2\mu) + b\mu R}{a\mu + b(1-2\mu)};$$

$$Q = \frac{a^2 b(1-\mu)^2 + 2a\mu(1-b)[2a\mu - b(1+\mu)]}{b^3(1-\mu)^2 - 2a(1-2\mu)(1-b)[2a\mu - b(1+\mu)]},$$

$$R = \frac{a^2 b(1-\mu)^2 + a\mu(1-b)[a\mu + b(1-2\mu)]}{b^3(1-\mu)^2 + a(2-\mu)(1-b)[a\mu + b(1-2\mu)]}.$$

With a large number of bond filler materials the values obtained from Eqs. (4) and (5) can be exploited for the elastic constants E and μ in the latter relations.

As regards the structural dimensions a and b, the application of the results gained from Eqs. (9) and (10) where P is the volume amount of the binder in the solid skeleton, while V_s means the volume proportion of this skeleton in the whole permeable system, appeared to be highly satisfactory:

$$a = \sqrt{\frac{PV_s}{3(1-b)}}, \tag{9}$$

$$b = \sqrt[3]{(1-P)V_s}. \tag{10}$$

Presenting the last relations concerning the elastic constants of the

real permeable system of a bond filler type, the restricted significance of the equations derived for the dry extreme state only must be pointed out. In order to formulate the corresponding relations for the elastic constants of the second extreme state, it is useful to return to Eqs. (1) and (2). This system is characterized by an irreversible content of a liquid in its free inner volume due to the extreme action of the internal surface forces. Since between an original empty phase and the skeleton the soft association could develop, the inherent elastic constants E_{f0}, μ_{f0} of this phase may be obtained from the H-D-N analogue of Eq. (1) as follows:

$$E_{f0} = \frac{(1-V_s)E\underline{E}_0}{E - \underline{E}_0}, \tag{11}$$

$$\mu_{f0} = \frac{\mu_0 E - \mu\underline{E}_0}{E - \underline{E}_0}. \tag{12}$$

The irreversibility of the clogging-up of the inner volume of a permeable system by a liquid can be guaranteed only by pure hand interactions between the free volume and the liquid contained, as well as between the apparent liquid skeleton and the original solid one. A repeated application of the Eq. (2) H-D-N analogue leads to the elastic constants, depending on the bulk modulus K_1 of the liquid, at first E_f, μ_f of the said liquid skeleton and finally on $\underline{E}$, $\underline{\mu}$ of the second boundary system considered:

$$E_f = E_{f0}\frac{E_{f0} + 3(1-2\mu_{f0})K_1}{E_{f0} + 2(1+\mu_{f0})(1-2\mu_{f0})K_1}, \tag{13}$$

$$\mu_f = \frac{\mu_{f0}E_{f0} + (1+\mu_{f0})(1-2\mu_{f0})K_1}{E_{f0} + 2(1+\mu_{f0})(1-2\mu_{f0})K_1}, \tag{14}$$

$$\underline{E} = \frac{[(1+\mu_f)V_s^2 E + (1+\mu)(1-V_s)E_f][(1-2\mu_f)V_s^2 E + (1-2\mu)(1-V_s)E_f]}{(1+\mu_f)(1-2\mu_f)V_s^2 E + (1+\mu)(1-2\mu)(1-V_s)E_f}, \tag{15}$$

$$\underline{\mu} = \frac{\mu(1+\mu_f)(1-2\mu_f)V_s^2 E + \mu_f(1+\mu)(1-2\mu)(1-V_s)E_f}{(1+\mu_f)(1-2\mu_f)V_s^2 E + (1+\mu)(1-2\mu)(1-V_s)E_f}. \tag{16}$$

Young's moduli $\underline{E}_0$, $\underline{E}$ and Poisson's ratios $\underline{\mu}_0$, $\underline{\mu}$ of the real permeable system of bond filler type, being dry and irreversibly full of liquid, respectively, represent the extreme values of the elastic constants of this system. Corresponding constants characterizing the elastic behaviour of moist bond filler composites in the ambient conditions at different humidity levels lie between these extreme values.

To illustrate the relations derived the courses of both the elastic constants of the binder-filler composites were plotted along the whole range of compositions; the diagrams also present a comparison with the values obtained by testing typical resin concretes (Fig. 4 (a) and (b)).
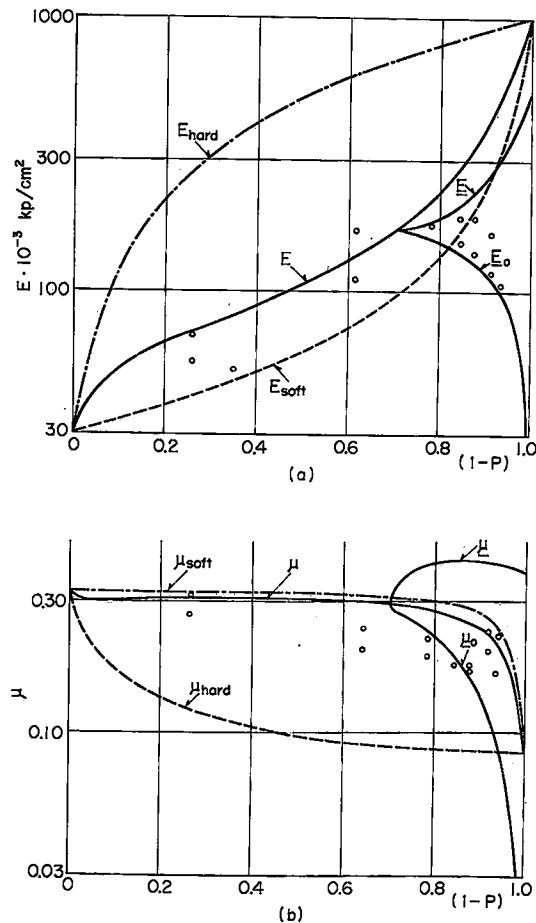
Fig. 4. Relationship of the (a) Young's moduli E and (b) Poisson's ratios μ of the binder-filler materials to the filler volume fraction $(1-P)$ of the total solid.

5. Elastance of a Material

The elasticity of a material is generally a function of time and the previous expressions are only valid exactly in the case of zero time of the history of the material and the phases or, if the constants of elasticity of the material components are known at any time, in the case of zero time of the history of the material only, for example, as a result of the time dependence on the contact stresses along the internal boundaries. In this sense the magnitudes derived in Sections 3 and 4 are to be con-

sidered as the principal (primary, initial) constants of elasticity.

Because the elasticity and other properties of the material are functions of time, it is necessary to find and define a new constant, based on the assessment of the energy situation in the system, in order to assess the material objectively. In our opinion, this approach can be accomplished by using the elastance of a material, *i.e.*, the measure of the ability of the material to be elastic over any period of time of its existence. This ability diminishes gradually, being exhausted by repeated loading or long-term loading, by temperature or humidity changes, etc. From this point of view, the elastance is to be understood as the basic qualitative characteristic of the elasticity of a material.

In order to evaluate the elastance quantitatively, a new constant of elasticity, called the absorbent of elasticity is now introduced. It is a function of the composition (the definition) of the material M and its history H, comprising the influence of time, temperature, sense and way of loading, stress state, etc., *i.e.*,

$$A = f(M, H). \tag{17}$$

In the previous paragraphs it was shown that the composition of a structured material is defined by its principal constants of elasticity, E and μ. Therefore,

$$A = F[(E, \mu), H]. \tag{18}$$

According to this definition, a single absorbent of elasticity corresponds to every defined material.

Its maximum exploitable value in a given system corresponds to the total energy of possible failure (destruction or dehesion energy); the material once strained or failed possesses a lower value of the absorbent of elasticity (a lower exploitable energetic reserve), the fully destructed material possesses a zero exploitable value. The decrease in the absorbent of elasticity during straining, which is a complementary value to the expended energy increase, varies from zero to the whole exploitable value.

A suitable measure of the elastance, for example, can be the ratio of the permanent and the elastic strains under external loading. The simultaneous measurement of the axial and the lateral strains in uniaxial loaded samples confirms that the composites have a different but characteristically limited absorbent of elasticity: the increased linear elastance in the axial direction corresponds to the decreasing of the latter in the lateral direction and vice versa (Fig. 5). The material behaves as if it could only accumulate (under certain loading) a certain, maximum amount of energy (manifested by a certain value of permanent strain). This amount can be absorbed sometimes preferring straining in the axial, sometimes in the lateral direction, probably in dependence on other factors not investigated (e.g., the initial state of stress, the com-
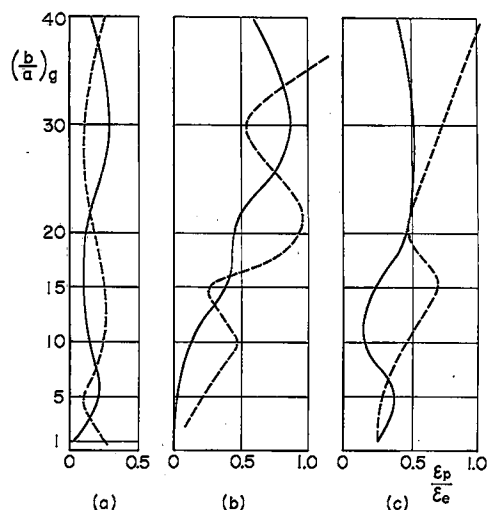
Fig. 5. Ratio of permanent ε_p and elastic ε_e strains in axial (solid line) and lateral (dotted line) directions, respectively, in proportion to the binder-filler weight relation. Stressing is one-third of the strength.

(a) epoxy resin concrete, (b) polyester resin concrete, (c) furanic resin concrete.

position of the system, etc.).

With increasing stress the elastance decreases in the same way the absorbent of elasticity decreases. This approach makes it possible to develop a concept of the final strength; it will be given by the integral of the energy consumption, characterized by the absorbent of elasticity. Because the absorbent of elasticity is a function of all conventional material characteristics, it can be taken as the fundamental parameter for their description.

The analysis of this constant and the relationship of the latter to conventional material characteristics doubtless needs further corroboration; nevertheless the prospective of the description of properties of every material by means of a single constant is a very attractive possibility and the authors are convinced of its reality.

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